Multilevel ensemble Kalman filtering

Håkon Hoel\textsuperscript{1} Kody Law\textsuperscript{2} Raúl Tempone\textsuperscript{3}

\textsuperscript{1}Department of Mathematics, University of Oslo, Norway

\textsuperscript{2}Oak Ridge National Laboratory, TN, USA

\textsuperscript{3}Applied Mathematics and Computational Sciences, KAUST university, Saudi Arabia

11th International EnKF Workshop, Ulvik
Overview

1. Problem description
2. Ensemble Kalman Filtering
3. Multilevel ensemble Kalman filtering
4. Numerical examples
5. Extension of MLEnKF and conclusion
Consider the underlying and unobservable dynamics

\[ u_{n+1} = u_n + \int_{n}^{n+1} a(u_t) \, dt + \int_{n}^{n+1} b(u_t) \, dW(t) =: \Psi(u_n) \]

with \( u_n \in \mathbb{R}^d \), and Lipschitz continuous \( a : \mathbb{R}^d \to \mathbb{R}^d \) and \( b : \mathbb{R}^d \to \mathbb{R}^{d \times \hat{d}} \).

And noisy observations

\[ y_n = H u_n + \gamma_n, \]

with i.i.d. \( \gamma \sim N(0, \Gamma) \) and \( H \in \mathbb{R}^{k \times d} \).

**Objective:** Let \( Y_n := (y_1, y_2, \ldots, y_n) \) and let \( Y_{n}^{obs} \) be a sequence of fixed observations. Construct an efficient method for tracking \( u_n|Y_n = Y_{n}^{obs} \).

That is, approximate

\[ \mathbb{E}\left[ \phi(u_n) \mid Y_n = Y_{n}^{obs} \right] \]

for an observable \( \phi : \mathbb{R}^d \to \mathbb{R} \).

**Abuse of notation:** will write \( u_n|Y_{n}^{obs} \) to represent \( u_n|Y_n = Y_{n}^{obs} \).
Problem description

Consider the underlying and unobservable dynamics

\[ u_{n+1} = u_n + \int_n^{n+1} a(u_t) \, dt + \int_n^{n+1} b(u_t) \, dW(t) =: \Psi(u_n) \]

with \( u_n \in \mathbb{R}^d \), and Lipschitz continuous \( a : \mathbb{R}^d \to \mathbb{R}^d \) and \( b : \mathbb{R}^d \to \mathbb{R}^{d \times \hat{d}} \).

And noisy observations

\[ y_n = Hu_n + \gamma_n, \]

with i.i.d. \( \gamma \sim N(0, \Gamma) \) and \( H \in \mathbb{R}^{k \times d} \).

**Objective:** Let \( Y_n := (y_1, y_2, \ldots, y_n) \) and let \( Y_n^{\text{obs}} \) be a sequence of fixed observations. Construct an efficient method for tracking \( u_n|_{(Y_n = Y_n^{\text{obs}})} \). That is, approximate

\[ \mathbb{E} \left[ \phi(u_n) \big| Y_n = Y_n^{\text{obs}} \right] \]

for an observable \( \phi : \mathbb{R}^d \to \mathbb{R} \).

Abuse of notation: will write \( u_n|_{Y_n^{\text{obs}}} \) to represent \( u_n|_{(Y_n = Y_n^{\text{obs}})} \).
Problem description

Consider the underlying and unobservable dynamics

\[
 u_{n+1} = u_n + \int_n^{n+1} a(u_t) \, dt + \int_n^{n+1} b(u_t) \, dW(t)
\]

\[=: \Psi(u_n) \]

with \( u_n \in \mathbb{R}^d \), and Lipschitz continuous \( a : \mathbb{R}^d \to \mathbb{R}^d \) and \( b : \mathbb{R}^d \to \mathbb{R}^{d \times \hat{d}} \).

And noisy observations

\[
y_n = Hu_n + \gamma_n,
\]

with i.i.d. \( \gamma \sim \mathcal{N}(0, \Gamma) \) and \( H \in \mathbb{R}^{k \times d} \).

Objective: Let \( Y_n := (y_1, y_2, \ldots, y_n) \) and let \( Y^{\text{obs}}_n \) be a sequence of fixed observations. Construct an efficient method for tracking \( u_n| (Y_n = Y^{\text{obs}}_n) \).

That is, approximate

\[
\mathbb{E} \left[ \phi(u_n) | Y_n = Y^{\text{obs}}_n \right]
\]

for an observable \( \phi : \mathbb{R}^d \to \mathbb{R} \).

Abuse of notation: will write \( u_n| Y^{\text{obs}}_n \) to represent \( u_n| (Y_n = Y^{\text{obs}}_n) \).
Overview

1 Problem description

2 Ensemble Kalman Filtering

3 Multilevel ensemble Kalman filtering

4 Numerical examples

5 Extension of MLEnKF and conclusion
Ensemble Kalman Filtering

Predict

1. Compute (numerical solutions of) $M$ particle paths one step forward

$$\hat{v}_{n+1,i} = \Psi(v_{n,i}, \omega_i) \text{ for } i = 1, 2, \ldots, M.$$ 

2. Compute sample mean and covariance

$$\hat{m}^{MC}_{n+1} = E_M[\hat{v}_{n+1}]$$
$$\hat{C}^{MC}_{n+1} = \text{Cov}_M[\hat{v}_{n+1}]$$

where

$$E_M[\hat{v}_{n+1}] := \frac{1}{M} \sum_{i=1}^{M} \hat{v}_{n+1,i}$$

and

$$\text{Cov}_M[\hat{v}_{n+1}] := E_M[\hat{v}_{n+1} \hat{v}_{n+1}^T] - E_M[\hat{v}_{n+1}](E_M[\hat{v}_{n+1}])^T.$$
Update

1. Generate signal observations for the ensemble of particles

   \[ \tilde{y}_{n+1,i} = y_{n+1}^{\text{obs}} + \gamma_{n+1,i} \quad \text{for } i = 1, 2 \ldots, M, \]

   with i.i.d. \( \gamma_{n+1,1} \sim \mathcal{N}(0, \Gamma) \).

2. Use signal observations to update particle paths

   \[ v_{n+1,i} = (I - K^{\text{MC}}_{n+1}H)\hat{v}_{n+1,i} + K^{\text{MC}}_{n+1}\tilde{y}_{n+1,i}, \]

   where \( K^{\text{MC}}_{n+1} = \hat{C}^{\text{MC}}_{n+1}H^T(\hat{C}^{\text{MC}}_{n+1}H^T + \Gamma)^{-1}. \)

Note: After the first step, all particles are correlated due to \( K^{\text{MC}}_{n+1} \).
From EnKF to mean field EnKF

For studying convergence properties of EnKF it is useful to introduce the mean field EnKF (MFEnKF)

$$
\begin{align*}
\hat{v}_{n+1,i}^{MF} &= \Psi(v_{n,i}^{MF}, \omega_i) \\
\hat{m}_{n+1}^{MF} &= E[\hat{v}_{n+1,i}^{MF}] \\
\hat{C}_{n+1}^{MF} &= \text{Cov}[\hat{v}_{n+1,i}^{MF}],
\end{align*}
$$

Up

$$
\begin{align*}
K_{n+1}^{MF} &= \hat{C}_{n+1}^{MF} H^T (H \hat{C}_{n+1}^{MF} H^T + \Gamma)^{-1} \\
\tilde{y}_{n+1,i}^{MF} &= y_{n+1}^{obs} + \gamma_{n+1,i} \\
v_{n+1,i}^{MF} &= (I - K_{n+1}^{MF} H) v_{n+1,i}^{MF} + K_{n+1}^{MF} \tilde{y}_{n+1,i}^{MF}.
\end{align*}
$$

and in comparison, EnKF

$$
\begin{align*}
\hat{v}_{n+1,i} &= \Psi(v_{n,i}, \omega_i) \\
\hat{m}_{n+1}^{MC} &= E_M[\hat{v}_{n+1}] \\
\hat{C}_{n+1}^{MC} &= \text{Cov}_M[\hat{v}_{n+1}],
\end{align*}
$$

Up

$$
\begin{align*}
K_{n+1}^{MC} &= \hat{C}_{n+1}^{MC} H^T (H \hat{C}_{n+1}^{MC} H^T + \Gamma)^{-1} \\
\tilde{y}_{n+1,i}^{MC} &= y_{n+1}^{obs} + \gamma_{n+1,i} \\
v_{n+1,i}^{MC} &= (I - K_{n+1}^{MC} H) \hat{v}_{n+1,i} + K_{n+1}^{MC} \tilde{y}_{n+1,i}.
\end{align*}
$$

- When underlying dynamics is linear with Gaussian additive noise and \( u_0 \) Gaussian, it holds that \( \mu_n^{MF}(dx) = P \left( u_n \in dx | Y_{n}^{obs} \right) \), where \( \mu_n^{MF} = \text{Law}(v_{n,i}^{MF}) \).
- In nonlinear settings, we use as approximation goal

$$
\int_{\mathbb{R}^d} \phi(x) \mu_n^{MF}(dx). \quad \text{NB!} \left( \mu_n^{MF} \neq P \left( u_n \in \cdot | Y_{n}^{obs} \right) \right).
$$
From EnKF to mean field EnKF

For studying convergence properties of EnKF it is useful to introduce the mean field EnKF (MFEnKF)

\[
\begin{aligned}
&\text{Pr} \begin{cases}
\hat{v}_{n+1,i}^\text{MF} = \psi(\hat{v}_{n,i}^\text{MF}, \omega_i) \\
\hat{m}_{n+1}^\text{MF} = \mathbb{E}[\hat{v}_{n+1,i}^\text{MF}] \\
\hat{C}_{n+1}^\text{MF} = \text{Cov}[\hat{v}_{n+1,i}^\text{MF}],
\end{cases} \quad \text{Up} \begin{cases}
K_{n+1}^\text{MF} = \hat{C}_{n+1}^\text{MF} H^T (H \hat{C}_{n+1}^\text{MF} H^T + \Gamma)^{-1} \\
\tilde{y}_{n+1,i} = y_{n+1}^\text{obs} + \gamma_{n+1,i} \\
v_{n+1,i}^\text{MF} = (I - K_{n+1}^\text{MF} H) \hat{v}_{n+1,i}^\text{MF} + K_{n+1}^\text{MF} \tilde{y}_{n+1,i}.
\end{cases}
\end{aligned}
\]

and in comparison, EnKF

\[
\begin{aligned}
&\text{Pr} \begin{cases}
\hat{v}_{n+1,i} = \psi(\hat{v}_{n,i}, \omega_i) \\
\hat{m}_{n+1}^\text{MC} = E_M[\hat{v}_{n+1}] \\
\hat{C}_{n+1}^\text{MC} = \text{Cov}_M[\hat{v}_{n+1}]
\end{cases} \quad \text{Up} \begin{cases}
K_{n+1}^\text{MC} = \hat{C}_{n+1}^\text{MC} H^T (H \hat{C}_{n+1}^\text{MC} H^T + \Gamma)^{-1} \\
\tilde{y}_{n+1,i} = y_{n+1}^\text{obs} + \gamma_{n+1,i} \\
v_{n+1,i} = (I - K_{n+1}^\text{MC} H) \hat{v}_{n+1,i} + K_{n+1}^\text{MC} \tilde{y}_{n+1,i}.
\end{cases}
\end{aligned}
\]

When underlying dynamics is linear with Gaussian additive noise and \( u_0 \) Gaussian, it holds that \( \mu_n^\text{MF}(dx) = \mathbb{P}\left(u_n \in dx \mid Y_{n}^\text{obs}\right) \), where \( \mu_n^\text{MF} = \text{Law}(v_{n,i}^\text{MF}) \).

In nonlinear settings, we use as approximation goal

\[
\int_{\mathbb{R}^d} \phi(x) \mu_n^\text{MF}(dx). \quad \text{NB!}(\mu_n^\text{MF} \neq \mathbb{P}\left(u_n \in \cdot \mid Y_{n}^\text{obs}\right) ).
\]
Theorem 1 (Le Gland et al. (2009))

Consider the dynamics and observations,

\[ u_{n+1} = f(u_n) + \xi_{n+1}, \quad \xi_{n+1} \sim N(0, \Sigma), \]
\[ y_{n+1} = H u_{n+1} + \gamma_{n+1}, \quad \gamma_{n+1} \sim N(0, \Gamma), \]

and assume \( u_0 \in L^p(\Omega) \) for any \( p \geq 1 \), and that

\[ \max(|f(x) - f(x')|, |\phi(x) - \phi(x')|) \leq C |x - x'| (1 + |x|^s + |x'|^s), \quad \text{for an } s \geq 0. \]

Then, for the EnKF update ensemble \( \{v_{n,i}\}_{i=1}^M \),

\[ \sup_{M \geq 1} \sqrt{M} \left( \mathbb{E} \left[ \left| \sum_{i=1}^M \frac{\phi(v_{n,i})}{M} - \int_{\mathbb{R}^d} \phi(x) \mu_n^{MF}(dx) \right|^p \right] \right)^{1/p} < \infty. \]

for any order \( p \geq 1 \) and finite \( n \).

Extension to further nonlinear settings in [Law et al. (2014)].
Computational cost of EnKF

To meet the constraint

\[
\left( \mathbb{E} \left[ \left| \sum_{i=1}^{M} \frac{\phi(v_{n,i})}{M} - \int_{\mathbb{R}^d} \phi(x) \mu_n^{MF}(dx) \right|^p \right] \right)^{1/p} = \mathcal{O}(\epsilon),
\]

one thus needs ensemble of size \( M = \mathcal{O}(\epsilon^{-2}) \).

How does the computational cost increase if the EnKF dynamics has to be sampled using a numerical solver for which

\[
|E[\Psi_{\Delta t} - \Psi]| = \mathcal{O}(\Delta t^\alpha)?
\]

Short answer (under additional assumptions): the cost increases to \( \mathcal{O}(\epsilon^{-(2+1/\alpha)}) \).
Overview

1 Problem description

2 Ensemble Kalman Filtering

3 Multilevel ensemble Kalman filtering

4 Numerical examples

5 Extension of MLEnKF and conclusion
Multilevel EnKF (MLEnKF)

Prediction

- Compute an ensemble of particle paths on a hierarchy of accuracy levels

\[
\hat{\nu}_{n+1,i}^{\ell-1} = \Psi_{\ell-1}^{\ell-1}(\nu_{n,i}^{\ell-1}, \omega_{\ell,i}), \quad \hat{\nu}_{n+1,i}^{\ell} = \Psi_{\ell}^{\ell}(\nu_{n,i}^{\ell}, \omega_{\ell,i}),
\]

for the levels \( \ell = 0, 1, \ldots, L \) and \( i = 1, 2, \ldots, M_{\ell} \).

- Multilevel approximation of mean and covariance matrices:

\[
\hat{m}_{n+1}^{\text{ML}} = \sum_{\ell=0}^{L} E_{M_{\ell}} \left[ \hat{\nu}_{n+1}^{\ell} - \hat{\nu}_{n+1}^{\ell-1} \right],
\]

\[
\hat{C}_{n+1}^{\text{ML}} = \sum_{\ell=0}^{L} \text{Cov}_{M_{\ell}} \left[ \hat{\nu}_{n+1}^{\ell} \right] - \text{Cov}_{M_{\ell}} \left[ \hat{\nu}_{n+1}^{\ell-1} \right]
\]

Notice the telescoping properties \( \mathbb{E} \left[ \hat{m}_{n+1}^{\text{ML}} \right] = \mathbb{E} \left[ \hat{\nu}_{n+1}^{L} \right] \) and
\[
\mathbb{E} \left[ \hat{C}_{n+1}^{\text{ML}} \right] = \text{Cov} \left( \hat{\nu}_{n+1}^{L} \right) + O(1/M_L).
\]
MLEnKF update step

**Update**

For $\ell = 0, 1, \ldots, L$ and $i = 1, 2, \ldots, M_\ell$,

\[
\tilde{y}_{n+1,i} = y_{n+1}^{obs} + \gamma_{n+1,i}, \quad \text{i.i.d.} \quad \gamma_{n+1,i} \sim \mathcal{N}(0, \Gamma)
\]

\[
\nu_{n+1,i}^{\ell-1} = (I - K_{n+1}^{ML} H)\hat{\nu}_{n+1,i}^{\ell-1} + K_{n+1}^{ML} \tilde{y}_{n+1,i},
\]

\[
\nu_{n+1,i}^{\ell} = (I - K_{n+1}^{ML} H)\hat{\nu}_{n+1,i}^{\ell} + K_{n+1}^{ML} \tilde{y}_{n+1,i},
\]

where

\[
K_{n+1}^{ML} = \hat{C}_{n+1}^{ML} H^T (H \hat{C}_{n+1}^{ML} H^T + \Gamma)^{-1}.
\]
For observables $\phi : \mathbb{R}^d \to \mathbb{R}$, introduce notation

$$
\mu_n^{ML}(\phi) := \sum_{\ell=0}^{L} \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} \phi(v_{n,i}^\ell) - \phi(v_{n,i}^{\ell-1}).
$$

and

$$
\mu_n^{MF}(\phi) := \int_{\mathbb{R}^d} \phi(x) \mu_n^{MF}(dx).
$$

**Question:** Under what assumptions and at what cost can one achieve

$$
\|\mu_n^{ML}(\phi) - \mu_n^{MF}(\phi)\|_{L^p(\Omega)} = O(\epsilon).
$$
Assumption 1

Consider the dynamics

\[ u_{n+1} = \Psi(u_n) = u_n + \int_n^{n+1} a(u_t)dt + \int_n^{n+1} b(u_t)dW(t), \quad n = 0, 1, \ldots \]

with \( u_0 \in \cap_{p \in \mathbb{N}} L^p(\Omega) \) and a hierarchy of numerical solvers \( \{\Psi^\ell\}_{\ell=0}^\infty \).

Furthermore, assume the observable \( \phi : \mathbb{R}^d \to \mathbb{R} \) satisfies

\[ |\phi(x) - \phi(x')| \leq C|x - x'|(1 + |x|^s + |x'|^s), \quad \text{for an } s \geq 0, \]

that there exists positive constants \( \alpha, \beta > 0 \), and an positive exponentially increasing sequence \( \{N_\ell\}_\ell \) such that for all \( u, v \in \cap_{p \in \mathbb{N}} L^p(\Omega) \),

(i) \(|\mathbb{E}[\phi(\Psi^\ell(u)) - \phi(\Psi^\ell(v))]| \lesssim N_\ell^{-\alpha}\), provided that \(|\mathbb{E}[u - v]| \lesssim N_\ell^{-\alpha}\),

(ii) \(\|\phi(\Psi^\ell(v)) - \phi(\Psi^{\ell-1}(v))\|_p \lesssim N_\ell^{-\beta}\), for all \( p \geq 1 \),

(iii) \(\text{Cost } (\Psi^\ell(v)) \lesssim N_\ell\).
Theorem 2 (MLEnKF accuracy vs. cost)

Suppose Assumption 1 holds. Then, for any $\epsilon > 0$ and $p \geq 2$, there exists an $L > 0$ and $\{M_\ell\}_{\ell=0}^L$ such that

$$\|\mu_n^\text{ML}(\phi) - \mu_n^\text{MF}(\phi)\|_p \lesssim \epsilon.$$  

And

$$\text{Cost (MLEnKF)} \lesssim \begin{cases} (|\log(\epsilon)|^{1-n}\epsilon)^{-2}, & \text{if } \beta > 1, \\ (|\log(\epsilon)|^{1-n}\epsilon)^{-2} |\log(\epsilon)|^3, & \text{if } \beta = 1, \\ (|\log(\epsilon)|^{1-n}\epsilon)^{-\left(2+\frac{1-\beta}{\alpha}\right)}, & \text{if } \beta < 1. \end{cases}$$  

(1)

In comparison,

$$\|\mu_n^\text{EnKF}(\phi) - \mu_n^\text{MF}(\phi)\|_p \lesssim \epsilon,$$

is achieved at cost $O\left(\epsilon^{-\left(2+\frac{1}{\alpha}\right)}\right)$. 
Theorem 2 (MLEnKF accuracy vs. cost)

Suppose Assumption 1 holds. Then, for any $\epsilon > 0$ and $p \geq 2$, there exists an $L > 0$ and $\{M_\ell\}_{\ell=0}^L$ such that

$$\|\mu_{n}^{\text{ML}}(\phi) - \mu_{n}^{\text{MF}}(\phi)\|_p \lesssim \epsilon.$$  

And

$$\text{Cost (MLEnKF)} \lesssim \begin{cases} (|\log(\epsilon)|^{1-n\epsilon})^{-2}, & \text{if } \beta > 1, \\ (|\log(\epsilon)|^{1-n\epsilon})^{-2} |\log(\epsilon)|^3, & \text{if } \beta = 1, \\ (|\log(\epsilon)|^{1-n\epsilon})^{-\left(2+\frac{1-\beta}{\alpha}\right)}, & \text{if } \beta < 1. \end{cases} \quad (1)$$

In comparison,

$$\|\mu_{n}^{\text{EnKF}}(\phi) - \mu_{n}^{\text{MF}}(\phi)\|_p \lesssim \epsilon,$$

is achieved at cost $O\left(\epsilon^{-\left(2+\frac{1}{\alpha}\right)}\right)$. 

Central idea in the proof

Introduce

$$\mu_{n}^{\text{MLMF}}(\phi) := \sum_{\ell=0}^{L} \frac{1}{M_{\ell}} \sum_{i=1}^{M_{\ell}} \phi(v_{n,\ell}^{\text{MF}}(\omega_{i,\ell})) - \phi(v_{n,\ell-1}^{\text{MF}}(\omega_{i,\ell}))$$

$$\mu_{n}^{\text{MF,L}}(\phi) := \mathbb{E}[\phi(v_{n}^{\text{MF,L}})] ,$$

and bound MLEnKF error by

$$\|\mu_{n}^{\text{ML}}(\phi) - \mu_{n}^{\text{MF}}(\phi)\|_{p} \leq \|\mu_{n}^{\text{ML}}(\phi) - \mu_{n}^{\text{MLMF}}(\phi)\|_{p}$$

$$+ \|\mu_{n}^{\text{MLMF}}(\phi) - \mu_{n}^{\text{MF,L}}(\phi)\|_{p} + \|\mu_{n}^{\text{MF,L}}(\phi) - \mu_{n}^{\text{MF}}(\phi)\|_{p}$$

$$\leq c \sum_{\ell=0}^{L} \left[ \|v_{n,\ell}^{\text{MF,L}} - v_{n,\ell}^{\text{MF}}\|_{\hat{p}} + \frac{\|v_{n,\ell}^{\text{MF,L}} - v_{n,\ell-1}^{\text{MF}}\|_{\hat{p}}}{M_{\ell}^{1/2}} \right] + \left| \mathbb{E}[\phi(v_{n}^{\text{MF,L}}) - \phi(v_{n}^{\text{MF}})] \right|$$

$$\leq c \left( \epsilon + \sum_{\ell=0}^{L} M_{\ell}^{-1/2} N_{\ell}^{-\beta/2} + N_{L}^{-\alpha} \right)$$
Overview

1 Problem description

2 Ensemble Kalman Filtering

3 Multilevel ensemble Kalman filtering

4 Numerical examples

5 Extension of MLEnKF and conclusion
Numerical example

Underlying dynamics is the Ornstein–Uhlenbeck SDE

\[ du = -udt + 0.5dW(t), \]

with a set of observations

\[ y_n = u_n + \gamma_n, \quad \text{i.i.d. } \gamma_n \sim N(0, 0.04) \]

Solvers: Hierarchy of Milstein solution operators \( \{ \Psi_\ell \}_{\ell=0}^L \) with \( \Delta t^\ell = O\left(2^{-\ell}\right) \).

Compare the approximation errors for the observable \( \phi(x) = x \) in terms of the RMSE

\[
\sqrt{\frac{N}{\sum_{n=1}^{N} \left| \mu_{n}^{\text{ML}}(\phi) - \mu_{n}^{\text{MF}}(\phi) \right|^2}}.
\]
Numerical example

Underlying dynamics is the Ornstein–Uhlenbeck SDE

\[ du = -u \, dt + 0.5 \, dW(t), \]

with a set of observations

\[ y_n = u_n + \gamma_n, \quad \text{i.i.d. } \gamma_n \sim N(0, 0.04) \]

Figure: From left to right: \( N = 100, 200 \) and 400.
Consider less regular observable $\phi(x) := 1\{x > 0.1\}$. Outside the scope of our theory since it does not hold that

$$\|\phi(\Psi^\ell(v)) - \phi(\Psi^{\ell-1}(v))\|_p \lesssim N_\ell^{-\beta}, \quad \forall p \geq 2.$$
Consider less regular observable $\phi(x) := 1\{x > 0.1\}$. Outside the scope of our theory since it does not hold that

$$\|\phi(\Psi^\ell(v)) - \phi(\Psi^{\ell-1}(v))\|_p \lesssim N^{-\beta}_\ell, \quad \forall p \geq 2.$$
Overview

1. Problem description
2. Ensemble Kalman Filtering
3. Multilevel ensemble Kalman filtering
4. Numerical examples
5. Extension of MLEnKF and conclusion
Extension of MLEnKF to infinite dimensional state spaces

- Work in progress with Alexey Chernov, Kody Law, Fabio Nobile and Tempone.
- Infinite dimensional stochastic dynamics:
  \[ u_{n+1} = \Psi(u_n) \]
  where \( u_n \in L^p(\Omega; \mathcal{H}) \) with \( \mathcal{H} = \text{Span}(\{\nu_i\}_{i=1}^{\infty}) \), and
  \( \Psi : L^p(\Omega; \mathcal{H}) \to L^p(\Omega; \mathcal{H}) \).
- And finite dimensional observations
  \[ y_n = Hu_n + \gamma_n, \]
  with linear \( H : \mathcal{H} \to \mathbb{R}^m \)
- Introduce nested hierarchy of Hilbert spaces
  \[ \mathcal{H}_0 \subset \mathcal{H}_1 \subset \ldots \subset \mathcal{H}_\infty = \mathcal{H}, \]
  where \( \mathcal{H}_\ell = \text{Span}(\{\nu_i\}_{i=1}^{N_\ell}) \) and work with a hierarchy of solvers
  \( \Psi^\ell : L^p(\Omega; \mathcal{H}_\ell) \to L^p(\Omega; \mathcal{H}_\ell) \).
### Conclusion

- Extended EnKF to multilevel EnKF.
- Verified asymptotic efficiency gain for approximations of expectation of observables. We hope to improve result further!
- Further extension of MLEnKF to infinite dimensional state space is work in progress.
